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1989 J. Phys. A: Math. Gen. 22 L231

(http://iopscience.iop.org/0305-4470/22/6/008)

J. Phys. A: Math. Gen. **22** (1989) L231–L236. Printed in the UK

LETTER TO THE EDITOR

On the spherical limit of anisotropic *n*-vector models

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Received 24 November 1988

Abstract. We show that, in an n-vector model with anisotropy affecting the interactions within a finite-dimensional subspace of the spin space, non-analyticity of the free energy can appear in the spherical limit ($n=\infty$) even in a finite system. The transition from the paramagnetic state is generically of mean-field type and is accompanied by spontaneous magnetisation within the anisotropy subspace.

In a recent paper (Angelescu *et al* 1986) the 'classical spin' model proposed by Fuller and Lenard (1979) has been analysed for lattice dimensions $d=1$ and $d=\infty$ (mean field). The area which proved to be of particular interest was the spherical limit of the model, where a phase transition appears in $d=1$, even for two interacting spins. Levine and Neuberger (1986) soon observed that the Fuller–Lenard (FL) model (in which the 'spins' are oriented two-dimensional planes in R^n, i.e. elements of the Grassman manifold $G(2, n)$) is very similar, especially for large n, to the CP^{n-1} field-theory model, studied in detail by di Vecchia *et al* (1984). However, there is no rigorous derivation of the spherical limit ($n\to\infty$) of the FL model in the general case (arbitrary d and external field) so it is unclear to what extent the equivalence with CP^{n-1} holds.

When trying to derive the $n\to\infty$ limit of the FL model we were led to consider such a limit for an *anisotropic n-vector model* with non-homogeneous interactions and external fields. In the isotropic case two methods have been conceived for dealing with non-homogeneous interactions.

(i) The coalescing bound method of Kac and Thompson (1971) was used by Knops (1973), but his derivation relies on a non-trivial assertion (see the argument outlined by Knops between equations (2.14) and (2.15) in his paper) which was left unproved.

(ii) The $1/n$ expansion (saddle-point) method of Abe (1973), rigorously developed in Angelescu *et al* (1979) and Kupiainen (1980), was used by Angelescu *et al* to study arbitrary finite ferromagnetic systems in the spherical limit and in particular the main theorem of Knops (1973) on the free energy has been rederived.

This letter is devoted to the study of the influence on the spherical limit of an anisotropy affecting a finite number of spin components in an otherwise isotropic n-vector model. Though we do not solve the problem at the level of complexity needed to handle the spherical limit of the FL model, we think the results throw some light on what is going on in that case, too.

Particular instances of anisotropic n-vector models with homogeneous interactions previously considered in the $n\to\infty$ limit are the 'm-vector-like' n-vector model with $m/n=\text{constant}\neq 0$, where m is the number of components affected by anisotropy (Suzuki 1973, Hikami and Abe 1974) and the extremely anisotropic (Ising-like) model

(Moore *et al* 1974, Aharony 1974, Hikami 1974, Pearce and Thompson 1976). In the latter case a mean-field behaviour has been found.

Our result shows that a mean-field transition is in a certain sense a 'generic' artefact of the spherical limit. The anisotropy has the role of simply shrinking the range of the spherical fields, which below T_c stick to a boundary. Hence, non-analiticity of the free energy may occur even for a finite number of spins and, if the anisotropy-induced boundary has several analytic pieces, several transitions are possible.

The model we study consists of N classical spins represented as $(n+k)$-dimensional vectors $(\tau_j, \sigma_j), j=1, \ldots, N, \tau_j \in R^n, \sigma_j \in R^k$, of length $(\tau_j^2+\sigma_j^2)^{1/2}=\sqrt{n+k}$, with interaction energy given by

$$\mathscr{H} = -\sum_{\mu=1}^{n} [\tfrac{1}{2}(\boldsymbol{\tau}^\mu, J\boldsymbol{\tau}^\mu)+(\boldsymbol{H}, \boldsymbol{\tau}^\mu)] - \sum_{\nu=1}^{k} [\tfrac{1}{2}(\boldsymbol{\sigma}^\nu, B^\nu\boldsymbol{\sigma}^\nu)+(\boldsymbol{h}^\nu, \boldsymbol{\sigma}^\nu)] \tag{1}$$

where $\boldsymbol{\tau}^\mu$, $\boldsymbol{\sigma}^\nu$, $\boldsymbol{H}$ and $\boldsymbol{h}^\nu$ are considered as vectors in R^N, e.g. $\boldsymbol{\sigma}^\nu=(\sigma_1^\nu, \ldots, \sigma_N^\nu)$ and brackets denote the usual scalar product in R^N, while J and B^ν are the interaction matrices. The second term in (1) accounts for an anisotropic interaction within the 'anisotropy subspace' R^k of the σ components.

We want to obtain the spherical limit of the free energy per spin component

$$F_N(\beta, \boldsymbol{H}, \boldsymbol{h}) = -\beta^{-1} \lim_{n\to\infty} (n+k)^{-1} \log Z_{N,n}(\beta, \boldsymbol{H}, \boldsymbol{h}) \tag{2}$$

where $Z_{N,n}$ is the partition function. For simplicity we assume ferromagnetism and positive external fields:

$$\begin{array}{llll} J_{ij} \geqslant 0 & B_{ij}^\nu \geqslant 0 & H_i \geqslant 0 & h_i^\nu \geqslant 0 \\ \forall_{i,j}=1, \ldots, N & & \text{and} & \forall \nu=1, \ldots, k. \end{array} \tag{3}$$

Moreover we make the (stronger than necessary) assumption that B^ν are connecting all sites, i.e. some power of the B^ν has no zero entry.

To compute $Z_{N,n}$ we account for the fixed spin-length condition using for $\delta(\tau_j^2+\sigma_j^2-n-k)$ the representation

$$\delta(x) = (\pi\beta)^{-1} \int \mathrm{d}t_j \exp[-\tfrac{1}{2}\beta(\gamma_j+\mathrm{i}t_j)x]$$

and perform the resulting Gaussian integrals over $\boldsymbol{\tau}^\mu$ and $\boldsymbol{\sigma}^\nu$. One gets

$$Z_{N,n} = (n^{k/2}/\pi\beta)^N \int_{R^N} \mathrm{d}\boldsymbol{t} \exp[nf_N(\hat{\boldsymbol{\gamma}}+\mathrm{i}\hat{\boldsymbol{t}}; \beta, \boldsymbol{H}, \boldsymbol{h})]$$
$$\times \prod_{\nu=1}^{k} \{\exp[\tfrac{1}{2}\beta\operatorname{Tr}(\hat{\boldsymbol{\gamma}}+\mathrm{i}\hat{\boldsymbol{t}})]/\det[(n\beta/2\pi)(\hat{\boldsymbol{\gamma}}+\mathrm{i}\hat{\boldsymbol{t}}-B^\nu)]\} \tag{4}$$

where $\hat{\boldsymbol{\gamma}}$ and $\hat{\boldsymbol{t}}$ are diagonal matrices with entries γ_j, t_j. Denoting $\boldsymbol{d}=\boldsymbol{\gamma}+\mathrm{i}\boldsymbol{t}$ we have

$$f_N(\boldsymbol{d}; \beta, \boldsymbol{H}, \boldsymbol{h}) = \tfrac{1}{2}\beta \operatorname{Tr} \hat{\boldsymbol{d}} - \tfrac{1}{2}\log\det[(\beta/2\pi)(\hat{\boldsymbol{d}}-J)] + \tfrac{1}{2}\beta((\hat{\boldsymbol{d}}-J)^{-1}\boldsymbol{H}, \boldsymbol{H})$$
$$+\tfrac{1}{2}\beta \sum_{\nu=1}^{k} [(\hat{\boldsymbol{d}}-B^\nu)^{-1}\boldsymbol{h}^\nu, \boldsymbol{h}^\nu]. \tag{5}$$

The representation (4) holds for all $\boldsymbol{\gamma}$ such that the Gaussian integrals converge, i.e. $\boldsymbol{\gamma} \in \mathscr{D}$ where

$$\mathscr{D} \equiv \{\boldsymbol{\gamma} \in R^N | \hat{\boldsymbol{\gamma}}-J>0, \hat{\boldsymbol{\gamma}}-B^\nu>0, \nu=1, \ldots, k\}. \tag{6}$$

Note that (4) is precisely of the form required to apply the saddle-point method. Clearly f_N is an analytic function of $\boldsymbol{d}$ on the domain $\mathrm{Re}\,\boldsymbol{d} \in \mathscr{D}$. One has to find the stationary points of f_N, i.e. to solve in this domain the system $\partial f_N/\partial d_j = 0, j = 1, \ldots, N$. The imaginary part of these equations has the structure $\Sigma_{j=1}^{N} A_{ij}(\gamma, t)t_j = 0$ where the matrix $A(\gamma, t) > 0$ for all $\boldsymbol{t}$ if $\boldsymbol{\gamma} \in \mathscr{D}$, implying that the only stationary points of f_N are real. This simplifies the matter considerably, because for $\boldsymbol{t} = 0$ f_N is a real, strictly convex function of $\boldsymbol{\gamma}$ on $\mathscr{D}$, so it can have at most one stationary point in $\mathscr{D}$, namely the point—if any—where f_N attains its minimum value.

Now, if $\boldsymbol{h}^\nu > 0$, $\forall \nu = 1, \ldots, k$, such a point surely exists in $\mathscr{D}$ because $f_N \to \infty$ whenever $\boldsymbol{\gamma} \to \infty$ or approaches the bounadary of $\mathscr{D}$. Indeed, if $\lambda_{\min}(\hat{\boldsymbol{\gamma}} - J) \to 0$ one has $-\log\det(\hat{\boldsymbol{\gamma}} - J) \to \infty$, while if $\lambda_{\min}(\hat{\boldsymbol{\gamma}} - B^\nu) \to 0$ then, due to our assumption (3) and the connectivity of B^ν, the corresponding eigenvector $\boldsymbol{e}^{(\nu)}(\boldsymbol{\gamma})$ has strictly positive components and hence non-zero projection onto $\boldsymbol{h}^\nu$, so

$$((\hat{\boldsymbol{\gamma}} - B^\nu)^{-1}\boldsymbol{h}^\nu, \boldsymbol{h}^\nu) \geqslant (\boldsymbol{h}^\nu, \boldsymbol{e}^{(\nu)}(\boldsymbol{\gamma}))^2/[\lambda_{\min}(\hat{\boldsymbol{\gamma}} - B^\nu)] \to \infty.$$

As a consequence the minimum point $\boldsymbol{\gamma}(\beta, \boldsymbol{H}, \boldsymbol{h})$ is itself analytic in all variables and is the unique solution in $\mathscr{D}$ of $\partial f_N/\partial\gamma_j = 0$, i.e.

$$\beta\left(1 - M_j^2 - \sum_{\nu=1}^{k} m_j^{(\nu)2}\right) = (\hat{\boldsymbol{\gamma}} - J)_{jj}^{-1} \qquad j = 1, \ldots, N \tag{7}$$

where the vectors $\boldsymbol{M}$ and $\boldsymbol{m}^{(\nu)}$ are defined by

$$(\hat{\boldsymbol{\gamma}} - J)\boldsymbol{M} = \boldsymbol{H} \qquad (\hat{\boldsymbol{\gamma}} - B^\nu)\boldsymbol{m}^{(\nu)} = \boldsymbol{h}^\nu \qquad \nu = 1, \ldots, k. \tag{8}$$

The free energy (2) given by the saddle-point method outlined above is the minimum value of f_N on $\mathscr{D}$:

$$F_N(\beta, \boldsymbol{H}, \boldsymbol{h}) = -f_N(\boldsymbol{\gamma}(\beta, \boldsymbol{H}, \boldsymbol{h}); \beta, \boldsymbol{H}, \boldsymbol{h}) \tag{9}$$

hence it is analytic. Thereby $\boldsymbol{M}$ and $\boldsymbol{m}^{(\nu)}$ are the magnetisations along axes μ and ν, i.e. the spherical limits of $\langle \boldsymbol{\tau}^\mu \rangle_n$ and $\langle \boldsymbol{\sigma}^\nu \rangle_n$ respectively. Moreover the whole asymptotic series ($1/n$ expansion) can be obtained in the usual way (Angelescu *et al* 1979).

On the other hand, if $\boldsymbol{h}^\nu = 0$ then f_N in (5) with $\boldsymbol{t} = 0$ no longer diverges when $\lambda_{\min}(\hat{\boldsymbol{\gamma}} - B^\nu) \to 0$ and it may happen that no stationary point exists in $\mathscr{D}$. Then it is much more difficult to obtain the asymptotics of (4), because one has to take into account the singularity structure (near the boundary) of the factors $\det(\hat{\boldsymbol{\gamma}} + \mathrm{i}\hat{\boldsymbol{t}} - B^\nu)^{-1}$. Fortunately, the dominant asymptotic contribution still depends only on the behaviour of f_N on

$$\tilde{\mathscr{D}} = \{\boldsymbol{\gamma} | \hat{\boldsymbol{\gamma}} - J > 0, \hat{\boldsymbol{\gamma}} - B^\nu \geqslant 0, \nu = 1, \ldots, k\} \tag{6$'$}$$

and one can obtain it by letting $\boldsymbol{h}^\nu \downarrow 0$ in the previous result. Thus

$$F_N(\beta, \boldsymbol{H}, 0) = -\min_{\boldsymbol{\gamma} \in \mathscr{D}} f_N(\boldsymbol{\gamma}; \beta, \boldsymbol{H}, 0) = -f_N(\boldsymbol{\gamma}(\beta, \boldsymbol{H})); \beta, \boldsymbol{H}, 0) \tag{10}$$

where $\boldsymbol{\gamma}(\beta, \boldsymbol{H}) = \lim_{\boldsymbol{h}^\nu \downarrow 0} \gamma(\beta, \boldsymbol{h}, \boldsymbol{h}^\nu)$. To find $\gamma(\beta, \boldsymbol{H})$ one has to solve (7) in $\tilde{\mathscr{D}}$ with $\boldsymbol{M}$ and $\boldsymbol{m}^{(\nu)}$ defined in (8) where we set $\boldsymbol{h}^\nu = 0$. The main consequence of this result is that $\boldsymbol{\gamma}(\beta, \boldsymbol{H})$, and hence F_N, may no longer be analytic and this happens when $\boldsymbol{\gamma}(\beta, \boldsymbol{H})$ pinches the boundary of $\tilde{\mathscr{D}}$. Looking at (8), one sees that in this case $\boldsymbol{m}^{(\nu)}$ may be non-zero and it will be an eigenvector of $\hat{\boldsymbol{\gamma}}(\beta, \boldsymbol{H}) - B^\nu$ corresponding to the zero eigenvalue. Thus one can obtain a phase transition in a finite system accompanied by spontaneous magnetisation in the anisotropy subspace.

To perform a more detailed study of such a transition we further simplify things by putting $\boldsymbol{H}=0$ as well. We then have to minimise on $\tilde{\mathscr{D}}$

$$f_N(\boldsymbol{\gamma};\beta,0,0)=\tfrac{1}{2}\beta\operatorname{Tr}\hat{\gamma}-\tfrac{1}{2}\log\det(\hat{\gamma}-J) \tag{11}$$

whose minimum point is denoted $\boldsymbol{\gamma}(\beta)$. Leaving aside the constraint $\hat{\gamma}-\boldsymbol{B}^{\nu}\geq 0$ that is minimising f_N on $\hat{\gamma}-J>0$, one obtains its minimum point $\boldsymbol{\gamma}^{(0)}(\beta)$ as the unique solution of

$$(\hat{\gamma}-J)^{-1}_{jj}=\beta \qquad j=1,\ldots,N. \tag{12}$$

For $\beta\downarrow 0$, $\gamma_j^{(0)}(\beta)\sim\beta^{-1}$, and therefore $\boldsymbol{\gamma}^{(0)}(\beta)\in\mathscr{D}$. For $\beta\to\infty$, $\boldsymbol{\gamma}^{(0)}(\beta)$ approaches the minimum point of $\operatorname{Tr}\hat{\gamma}$ on $\lambda_{\min}(\hat{\gamma}-J)=0$, which is $\gamma_j^{(0)}(\infty)=\Sigma_{i=1}^{N}J_{ij}$. A transition will appear if and only if $\boldsymbol{\gamma}^{(0)}(\beta)$ crosses the boundary of $\mathscr{D}$, which happens if and only if $\hat{\boldsymbol{\gamma}}^{(0)}(\infty)-\boldsymbol{B}^{\nu}$ is not positive semidefinite for some ν. If the latter condition is fulfilled, let β_c be the supremum of β for which $\boldsymbol{\gamma}^{(0)}(\beta)\in\mathscr{D}$ and let $\boldsymbol{\gamma}_c$ be $\boldsymbol{\gamma}^{(0)}(\beta_c)$. Then, for at least one ν, $\hat{\boldsymbol{\gamma}}_c-\boldsymbol{B}^{\nu}$ has the zero eigenvalue (while still positive semidefinite).

We shall study the transition at β_c in the generic case when:

(i) for only one ν, say $\nu=1$, is $\hat{\boldsymbol{\gamma}}_c-\boldsymbol{B}^{\nu}$ singular while $\hat{\boldsymbol{\gamma}}_c-\boldsymbol{B}^{\nu}>0$ for $\nu\neq 1$;

(ii) the curve $\boldsymbol{\gamma}^{(0)}(\beta)$ is transverse to the manifold $\lambda_{\min}(\hat{\gamma}-\boldsymbol{B}^1)=0$, i.e.

$$\left(\frac{\mathrm{d}\boldsymbol{\gamma}^{(0)}}{\mathrm{d}\beta}(\beta_c),\operatorname{grad}_{\gamma}(\lambda_{\min}(\hat{\gamma}_c-\boldsymbol{B}^1))\right)<0. \tag{13}$$

Now an easy computation shows that $\operatorname{grad}_{\gamma}\lambda_{\min}(\hat{\gamma}_c-\boldsymbol{B}^1)=\boldsymbol{e}^{(1)2}$ where $\boldsymbol{e}^{(1)2}$ is the vector whose components are $(e_j^{(1)})^2$ and $\boldsymbol{e}^{(1)}$ is the normalised eigenvector of $\hat{\boldsymbol{\gamma}}_c-\boldsymbol{B}^1$ with zero eigenvalue. Also

$$\frac{\mathrm{d}\boldsymbol{\gamma}^{(0)}(\beta)}{\mathrm{d}\beta}=-P(\boldsymbol{\gamma}^{(0)}(\beta))\mathbf{1} \tag{14}$$

where $\mathbf{1}$ is the vector with all components equal to 1 and $[P(\boldsymbol{\gamma})^{-1}]_{ij}\equiv[(\hat{\gamma}-J)^{-1}_{ij}]^2$. Thus (13) becomes

$$(P(\boldsymbol{\gamma}_c)\mathbf{1},\mathbf{e}^{(1)2})>0. \tag{13$'$}$$

For $\beta>\beta_c$ in a neighbourhood of β_c, $\boldsymbol{\gamma}(\beta)$ will belong to the manifold $\lambda_{\min}(\hat{\gamma}-\boldsymbol{B}^1)=0$. By (3) and the connectivity of $\boldsymbol{B}^1$, $\lambda_{\min}(\hat{\gamma}-\boldsymbol{B}^1)$ is non-degenerate and its eigenvector $\boldsymbol{e}^{(1)}(\boldsymbol{\gamma})$ has strictly positive components. In particular, the manifold $\lambda_{\min}(\hat{\gamma}-\boldsymbol{B}^1)=0$ is analytic hence $\boldsymbol{\gamma}(\beta)$ is analytic for $\beta>\beta_c$.

The critical behaviour is controlled by the behaviour of $\boldsymbol{\gamma}(\beta)$ near β_c. For $\beta<\beta_c$, $\boldsymbol{\gamma}(\beta)=\boldsymbol{\gamma}^{(0)}(\beta)$ and the relevant information is provided by (14). For $\beta>\beta_c$ one accounts for $\lambda_{\min}(\hat{\gamma}-\boldsymbol{B}^1)=0$ by the Lagrange multiplier method, and obtains a jump of $\mathrm{d}\boldsymbol{\gamma}(\beta)/\mathrm{d}\beta$ at β_c

$$\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}\beta}(\beta_c^+)-\frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}\beta}(\beta_c^-)=\rho P(\boldsymbol{\gamma}_c)\boldsymbol{e}^{(1)2} \tag{15}$$

where

$$\rho=(P(\boldsymbol{\gamma}_c)\mathbf{1},\boldsymbol{e}^{(1)2})/(P(\boldsymbol{\gamma}_c)\boldsymbol{e}^{(1)2},\boldsymbol{e}^{(1)2})>0 \tag{16}$$

is the derivative of the Lagrange multiplier at β_c^+.

The *spontaneous magnetisation* will, for $\beta>\beta_c$ and $\boldsymbol{m}^{(1)}>0$, be proportional to $\boldsymbol{e}^{(1)}(\boldsymbol{\gamma}(\beta))$. By taking the β derivative in (7) one gets for $\beta\downarrow\beta_c$

$$\frac{\mathrm{d}\boldsymbol{m}^{(1)2}}{\mathrm{d}\beta}(\beta_c^+)=\frac{\rho}{\beta_c}\boldsymbol{e}^{(1)2}$$

From which, by integration, and setting as usual $t=(\beta-\beta_c)/\beta_c$,

$$m_j^{(1)}(\beta)\sim t^{1/2}\rho^{1/2}e_j^{(1)} \qquad t\downarrow 0 \tag{17}$$

i.e. the critical index of $\boldsymbol{m}^{(1)}$ is $\frac{1}{2}$.

The *susceptibility matrix* $\chi_{ij}=\partial m_i^{(1)}/\partial h_j^1$ is given by

$$\chi(\beta)=(\hat{\gamma}-B^1+2\beta\hat{\boldsymbol{m}}^{(1)}P(\boldsymbol{\gamma})\hat{\boldsymbol{m}}^{(1)})^{-1}. \tag{18}$$

When $\beta\uparrow\beta_c$, $\boldsymbol{m}^{(1)}=0$, $\boldsymbol{\gamma}(\beta)=\boldsymbol{\gamma}^{(0)}(\beta)$ and

$$-\lambda_{\min}(\hat{\boldsymbol{\gamma}}^{(0)}(\beta)-B^1)\sim t\beta_c\frac{\mathrm{d}\lambda_{\min}}{\mathrm{d}\beta}(\beta_c^-)=-\beta_c t(P(\boldsymbol{\gamma}_c)\mathbf{1},\boldsymbol{e}^{(1)2})$$

from which we obtain

$$\chi_{ij}(\beta)\sim e_i^{(1)}e_j^{(1)}/\lambda_{\min}(\hat{\boldsymbol{\gamma}}^{(0)}(\beta)-B^1)\sim|t|^{-1}[\beta_c(P(\boldsymbol{\gamma}_c)\mathbf{1},\boldsymbol{e}^{(1)2})]^{-1}e_i^{(1)}e_j^{(1)}. \tag{19}$$

For $\beta>\beta_c$, equation (18) still makes sense because $\hat{\boldsymbol{m}}^{(1)}\boldsymbol{P}\hat{\boldsymbol{m}}^{(1)}>0$ and the minimum eigenvalue of $\hat{\boldsymbol{\gamma}}-B^1+2\beta\hat{\boldsymbol{m}}^{(1)}P(\boldsymbol{\gamma})\hat{\boldsymbol{m}}^{(1)}$ behaves for $t\downarrow 0$ like the diagonal element of $2\beta\hat{\boldsymbol{m}}^{(1)}\boldsymbol{P}\hat{\boldsymbol{m}}^{(1)}$. Hence

$$\chi_{ij}(\beta)\sim t^{-1}[2\beta_c(P(\boldsymbol{\gamma}_c)\mathbf{1},\boldsymbol{e}^{(1)2})]^{-1}e_i^{(1)}e_j^{(1)} \tag{20}$$

and therefore the susceptibility critical index equals 1, from both sides.

Finally, the specific heat in zero external field

$$C(\beta)=-\beta^2\frac{\partial^2(\beta F_N)}{\partial\beta^2}=\frac{N}{2}+\frac{\beta^2}{2}\frac{\mathrm{d}}{\mathrm{d}\beta}\operatorname{Tr}\hat{\gamma}(\beta). \tag{21}$$

Then (15) shows that there is a positive specific heat jump at β_c,

$$\Delta C=\tfrac{1}{2}\rho\beta_c^2. \tag{22}$$

Thus, the transition at β_c from the disordered high-temperature state to a state with spontaneous magnetisation in the anisotropy subspace has, under assumption (13), a mean-field character.

When β is further increased, $\boldsymbol{\gamma}(\beta)$ will move within the boundary of $\mathscr{D}$ and a new transition may occur when, e.g., $\hat{\boldsymbol{\gamma}}(\beta)-B^2$ becomes singular. We exemplify this by producing an example of a three-spin system sufficiently simple to allow analytic calculations. The isotropic part of the interaction and the external fields are taken zero. The interaction matrices for the first two spin components are

$$B^1=\begin{pmatrix}0&0&0\\0&0&5\\0&5&0\end{pmatrix} \qquad B^2=\begin{pmatrix}0&3&0\\3&0&3\\0&3&0\end{pmatrix}. \tag{23}$$

We have

$$f_N(\boldsymbol{\gamma})=\frac{1}{2}\sum_{j=1}^{3}(\beta\gamma_j-\ln\gamma_j)+\text{constant}. \tag{24}$$

Hence $\gamma_j^{(0)}(\beta)=\beta^{-1}$, which is in $\mathscr{D}$ for $\beta<\beta_c=\frac{1}{5}$. For $\beta\in(\frac{1}{5},\frac{16}{45})$, $\boldsymbol{\gamma}(\beta)=(1/\beta,\frac{1}{5},\frac{1}{5})$, for which $\lambda_{\min}(\hat{\boldsymbol{\gamma}}(\beta)-B^1)=0$ and $\hat{\boldsymbol{\gamma}}(\beta)-B^2>0$. In this interval $\boldsymbol{m}^{(2)}(\beta)=(0,0,0)$ while $\boldsymbol{m}^{(1)}(\beta)=[1-(\beta_c/\beta)]^{1/2}\,(0,1,1)$. At $\beta=\beta_c'=\frac{16}{45}$, $\lambda_{\min}(\hat{\boldsymbol{\gamma}}(\beta_c')-B^2)=0$ too. When $\beta>\beta_c'$, $(\hat{\boldsymbol{\gamma}}(\beta)-B^1)\boldsymbol{m}^{(1)}=0$, $(\hat{\boldsymbol{\gamma}}(\beta)-B^2)\boldsymbol{m}^{(2)}=0$, $\boldsymbol{m}^{(1)}$, $\boldsymbol{m}^{(2)}>0$, from which one can express $\boldsymbol{\gamma}$ only in terms of $x=(m_2^{(1)}/m_3^{(1)}$:

$$\boldsymbol{\gamma}(\beta)=\left(\frac{x}{\beta_c'},\frac{1}{\beta_c x},\frac{x}{\beta_c}\right).$$

Introducing this into (7), one obtains

$$m_1^{(1)}=0 \qquad m_2^{(1)}=\frac{\sqrt{7}}{4} \qquad m_3^{(1)}=\frac{\sqrt{7}}{4x}$$

$$m_1^{(2)}=\left(1-\frac{\beta_c'}{\beta x}\right)^{1/2} \qquad m_2^{(2)}=\frac{5x\beta_c}{3\beta_c'}\left(1-\frac{\beta_c'}{\beta x}\right)^{1/2} \qquad m_3^{(2)}=\frac{x\beta_c}{\beta_c'}\left(1-\frac{\beta_c'}{\beta x}\right)^{1/2}$$

where x is the positive root of $(\frac{5}{4})^2x^2-(1/5\beta)x-1=0$. Thus, beyond β_c', the spontaneous magnetisation changes its direction in the anisotropy subspace and, for $\beta\to\infty$, approaches its saturation value 1.

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